

# Deterministic polynomial-time test for prime ideals in a Dedekind domain with finite rank

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## Abstract

We describe a deterministic polynomial-time test that determining whether a nonzero ideal is a prime ideal in a Dedekind domain with finite rank. The techniques which we used are basis representation of finite rings and the Hermite and Smith normal forms.

## 1 Introduction

Primality testing has been proved to be in P by Agrawal, Kayal and Saxena [1] in 2004. From an algebraic point of view, a deterministic polynomial-time algorithm for testing the primality of ideals in the ring of integers  $\mathbb{Z}$  can be derived from the AKS test. It is natural to ask whether a deterministic polynomial-time algorithm exists for testing whether an ideal is a prime ideal in a general ring. In this paper, we will give an affirmative answer for all Dedekind domains with finite rank, where finite rank means that the ring as a  $\mathbb{Z}$ -module is finitely generated. Moreover, we present a deterministic polynomial-time algorithm for testing whether a nonzero ideal is a power of a prime ideal in such a Dedekind domain with finite rank.

Cohen [4, 5] demonstrated an algorithm which can be used for testing the primality of nonzero ideals in the ring of algebraic integers of a number field. The algorithm mainly consists of two steps. The first is to compute the prime ideal factorization of an associated prime number in the algebraic integers ring. The second is to compute the corresponding valuation of the tested ideal at each prime ideal of the first step. Actually, the algorithm of [4, 5] is to determine the primality of an ideal by decomposing this ideal in the algebraic integers ring. Factoring univariate polynomials over finite fields is used in the first step of this algorithm. It is known that a deterministic polynomial-time

algorithm for factoring polynomials over finite fields does not exist so far. Hence the algorithm of [4, 5] can not be deterministic and polynomial-time at the same time.

The idea of this paper is to study the ring structure of the factor rings of nonzero ideals in a Dedekind domain of finite rank. A proper ideal is a prime ideal if the corresponding factor ring is a field. Therefore, it is natural to consider the factor ring. The main contribution of this paper is to compute basis representation of the factor rings, which is a bridge between the characteristic of the factor rings and the primality testing of ideals. We will apply the algorithms of [6] by Hafner and McCurley for computing Hermite and Smith normal forms to obtain the required basis representation.

The concept of basis representation is to characterize finite rings. In 1992, this notion was first proposed by Lenstra [11] to describe finite fields. In 2006, Kayal and Saxena [10] stated the formal definition of basis representation for finite rings. At the same year, Arvind, Das and Mukhopadhyay [2] proved that field testing for finite rings in basis representation is in P. Their result gave us great confidence on prime ideal testing. We will show the asymptotical time bound of their algorithm of field testing in this paper. In 2013, Staromiejski [15] presented a deterministic polynomial-time algorithm for local ring testing, which inspires our another algorithm of prime ideal power testing directly. Additionally, the running time of our algorithms for testing prime ideals and prime ideal powers will be decreased if we turn to employ a randomized primality test, as [15] pointed out.

The paper is organized as follows. In Section 2 we introduce some facts and algorithms on Dedekind domains, Hermite and Smith normal forms, which will be needed in Section 3. The main issues of this paper are raised in Section 3. And deterministic polynomial-time algorithms for testing prime ideals and prime ideal powers in a Dedekind domain with finite rank are discussed. Section 4 is devoted to conclusions on the relevant analysis and comprehension of the running time of our algorithms.

Throughout the paper, all rings are assumed to be commutative and with multiplicative identity, written as 1, and  $1 \neq 0$ . We denote by  $M(t)$  a upper bound for the number of bit operations required to multiply two  $[t]$  bit integers. By a result of Schönhage and Strassen [12],  $M(t) = O(t \log t \log \log t)$ . Similarly, by  $B(t)$  we denote the number of bit operations of the operation which is the application of the Chinese remainder theorem with moduli consisting of all primes less than  $t$ . We can take  $B(t) = O(M(t) \log t)$  from [6]. We denote by  $\omega$  the exponent for matrix multiplication, and  $2 < \omega \leq 3$ .

## 2 Preliminaries

We begin with recall the definition and some properties of Dedekind domains, which can be found in the book by Janusz [9].

**Definition 2.1.** *A ring  $\mathcal{O}$  is a Dedekind domain if it is a noetherian integral domain*

such that the localization  $\mathcal{O}_{\mathfrak{p}}$  is a discrete valuation ring for every nonzero prime ideal  $\mathfrak{p}$  of  $\mathcal{O}$ .

**Proposition 2.2.** *Let  $\mathcal{O}$  be a Dedekind domain. Let  $I, J$  be nonzero ideals of  $\mathcal{O}$ . Then the following assertions hold:*

- (1) *Every nonzero prime ideal of  $\mathcal{O}$  is a maximal ideal.*
  - (2) *The ideal  $I$  can be uniquely written as a product of prime ideals.*
  - (3) *For any  $\alpha \in I$  with  $\alpha \neq 0$  there exists  $\beta \in I$  such that  $I = (\alpha, \beta) = \mathcal{O}\alpha + \mathcal{O}\beta$ .*
- Moreover, suppose  $\mathcal{O}$  is of finite rank, and the rank of  $\mathcal{O}$  is  $n$ . Then*
- (4) *The ideal  $I$  is a finitely generated free  $\mathbb{Z}$ -module with the same rank  $n$  as  $\mathcal{O}$ , and the factor ring  $\mathcal{O}/I$  is a finite ring.*

*We define the norm of  $I$ ,  $\mathcal{N}(I)$ , to be the order of  $\mathcal{O}/I$ , i.e.  $\mathcal{N}(I) = |\mathcal{O}/I|$ .*

- (5) *(Property of the norm)  $\mathcal{N}(IJ) = \mathcal{N}(I)\mathcal{N}(J)$ .*

*We denote  $\mathcal{N}(\alpha) = \mathcal{N}(I)$  when  $I = \mathcal{O}\alpha$  is a principal ideal.*

Combining the results of commutative algebras [3] with Dedekind domains, we obtain the following results which are needed in later context.

**Proposition 2.3.** *Let  $\mathcal{O}$  be a Dedekind domain. Let  $I$  be a nonzero proper ideal of  $\mathcal{O}$ . Then  $I$  is a prime ideal if and only if  $\mathcal{O}/I$  is a field.*

*Proof* The assertion follows from Proposition 2.2 (1) directly. □

**Proposition 2.4.** *Let  $\mathcal{O}$  and  $I$  be as Proposition 2.3. Then  $I$  is a power of a prime ideal if and only if  $\mathcal{O}/I$  is a local ring.*

*Proof* If  $I = \mathfrak{p}^k$ ,  $\mathfrak{p}$  is a nonzero prime ideal and  $k > 0$ , then  $\mathfrak{p}/\mathfrak{p}^k$  is the unique maximal ideal of  $\mathcal{O}/I = \mathcal{O}/\mathfrak{p}^k$ , hence  $\mathcal{O}/I$  is a local ring. Conversely, suppose  $I$  is not a prime ideal power. There must exist two distinct prime ideals  $\mathfrak{p}$  and  $\mathfrak{q}$  such that  $I \subseteq \mathfrak{p}\mathfrak{q}$  by Proposition 2.2 (2). Therefore  $\mathfrak{p}/I$  and  $\mathfrak{q}/I$  are different maximal ideals in the factor ring  $\mathcal{O}/I$ . It contradicts the assumption that  $\mathcal{O}/I$  is a local ring. □

Next we introduce the Hermit and Smith normal forms [7, 14] of matrices over  $\mathbb{Z}$ . Let  $A$  be an  $n \times m$  matrix with integer entries,  $B$  be an  $n \times n$  nonsingular integer matrix. For the definitions of Hermit normal form (abbreviated HNF)  $H$  of  $A$  and Smith normal form (abbreviated SNF)  $S$  of  $B$  of this paper, we refer to the book by Cohen [4]. Moreover,  $H$  and  $S$  are uniquely existed for  $A$  and  $B$  (see [4, Chapter 2, Theorems 2.4.3 and 2.4.12]). There are many algorithms for computing the HNF and SNF of matrices. What we need are the following algorithms that are derived from the results of Hafner and McCurley [6] directly. And the method of [8] by Iliopoulos is also used to obtain Proposition 2.6.

**Proposition 2.5.** *There exists a deterministic algorithm that receives as input an  $n \times m$  integral matrix  $A$  of rank  $n$  and a positive integer  $h$  that is a multiple of  $\det(\mathcal{L}(A))$ , and produces as output the HNF  $H$  of  $A$  such that  $AU = H$ , where  $U$  is an  $m \times m$  unimodular matrix. The running time of the algorithm is  $O(mnB(\log T) + mn^2B(\log h))$  bit operations, if the entries of  $A$  are bounded in absolute value by  $T$ .*

**Proposition 2.6.** *There exists a deterministic algorithm that receives as input an  $n \times n$  nonsingular integral matrix  $B$  and a positive integer  $h$  that is a multiple of  $\det(B)$ , and produces as output the SNF  $S$  of  $B$  and the transforming matrices  $U, V$  such that  $VBU = S$ , where  $U, V$  are  $n \times n$  unimodular matrices. The running time of the algorithm is  $O(n^2B(\log T) + n^3B(\log h)\log h)$  bit operations, if the entries of  $B$  are bounded in absolute value by  $T$ .*

Now we can apply HNF to a Dedekind domain  $\mathcal{O}$  of finite rank and its nonzero ideal  $I$ . We may assume  $\mathcal{O} = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n$  and  $I = \mathbb{Z}\beta_1 \oplus \dots \oplus \mathbb{Z}\beta_n$  as  $\mathbb{Z}$ -modules. There exists a matrix  $A \in M_n(\mathbb{Z})$  such that

$$(\beta_1, \dots, \beta_n) = (\omega_1, \dots, \omega_n)A.$$

Let  $H$  be the HNF of  $A$ , such that  $H = AU$  and  $U \in \text{GL}_n(\mathbb{Z})$ . We denote a vector

$$(\gamma_1, \dots, \gamma_n) = (\omega_1, \dots, \omega_n)H = (\beta_1, \dots, \beta_n)U.$$

Since  $U$  is an unimodular matrix, we get

$$I = \mathbb{Z}\beta_1 \oplus \dots \oplus \mathbb{Z}\beta_n = \mathbb{Z}\gamma_1 \oplus \dots \oplus \mathbb{Z}\gamma_n.$$

And we have  $\mathcal{N}(I) = \det(H) = |\det(A)|$ . Here we call  $\{\gamma_1, \dots, \gamma_n\}$  the HNF basis of  $I$  with respect to  $\mathbb{Z}$ -basis  $\{\omega_1, \dots, \omega_n\}$  of  $\mathcal{O}$ .

Continue to apply SNF to the previous  $\mathcal{O}$  and  $I$ , we gain the following proposition, which originated in [4, Chapter 2, Theorem 2.4.13].

**Proposition 2.7.** *Let  $\mathcal{O}$  be a Dedekind domain with finite rank,  $I$  is a nonzero ideal of  $\mathcal{O}$ . Then there exist positive integers  $d_1, \dots, d_n$  satisfying the following conditions:*

- (1) *For every  $i$  such that  $1 \leq i < n$  we have  $d_{i+1} | d_i$ .*
- (2) *We have an isomorphism as  $\mathbb{Z}$ -modules*

$$(\mathcal{O}/I, +) \cong \bigoplus_{1 \leq i \leq n} (\mathbb{Z}/d_i\mathbb{Z}, +)$$

*and in particular  $\mathcal{N}(I) = \prod_{i=1}^n d_i$ .*

- (3) *There exists a  $\mathbb{Z}$ -basis  $\{\alpha_1, \dots, \alpha_n\}$  of  $\mathcal{O}$  such that  $\{d_1\alpha_1, \dots, d_n\alpha_n\}$  is a  $\mathbb{Z}$ -basis of  $I$ .*

*Furthermore, the  $d_i$  are uniquely determined by  $\mathcal{O}$  and  $I$ .*

*Proof* We may suppose  $\mathcal{O} = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n$  and  $I = \mathbb{Z}\beta_1 \oplus \dots \oplus \mathbb{Z}\beta_n$ . Thus we have

$$(\beta_1, \dots, \beta_n) = (\omega_1, \dots, \omega_n)A$$

where  $A \in M_n(\mathbb{Z})$  and  $\det(A) \neq 0$ . Let  $S$  be the SNF of  $A$  such that  $S = VAU$ ,  $U, V \in GL_n(\mathbb{Z})$ , and

$$S = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{pmatrix}$$

where  $d_{i+1} | d_i$ ,  $1 \leq i < n$ . We denote a vector

$$(\alpha_1, \dots, \alpha_n) = (\omega_1, \dots, \omega_n)V^{-1}$$

Hence we get

$$(d_1\alpha_1, \dots, d_n\alpha_n) = (\alpha_1, \dots, \alpha_n)S = (\omega_1, \dots, \omega_n)AU = (\beta_1, \dots, \beta_n)U.$$

And  $\mathcal{O} = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n$ ,  $I = \mathbb{Z}(d_1\alpha_1) \oplus \dots \oplus \mathbb{Z}(d_n\alpha_n)$  because of  $V^{-1}$ ,  $U \in GL_n(\mathbb{Z})$ . Thereby as  $\mathbb{Z}$ -modules:

$$(\mathcal{O}/I, +) \cong \bigoplus_{1 \leq i \leq n} (\mathbb{Z}/d_i\mathbb{Z}, +).$$

The uniqueness of  $d_i$  can be verified by showing their independence of the choice of the initial  $\mathbb{Z}$ -basis of  $\mathcal{O}$  and  $I$ .  $\square$

### 3 Tests for prime ideals and prime ideal powers

In this section we will present our algorithms of testing prime ideals and prime ideal powers in a Dedekind domain with finite rank. We characterize them by the following problems.

**Problem 3.1.** *Given a multiplication table of a  $\mathbb{Z}$ -basis  $\{\omega_1, \dots, \omega_n\}$  of a Dedekind domain  $\mathcal{O}$  with finite rank, determine the primality of an arbitrary nonzero ideal  $I$  of  $\mathcal{O}$ .*

**Problem 3.2.** *Given a multiplication table of a  $\mathbb{Z}$ -basis  $\{\omega_1, \dots, \omega_n\}$  of a Dedekind domain  $\mathcal{O}$  with finite rank, determine whether  $I$  is a power of a prime ideal in  $\mathcal{O}$ , where  $I$  is an arbitrary nonzero ideal of  $\mathcal{O}$ .*

For simplicity we denote  $\{\omega_1, \dots, \omega_n\}$  by  $\mathcal{W}$  in this section. The multiplication table of  $\mathcal{W}$  is a sequence of integers  $((c_{ijk})_{i,j,k=1,\dots,n})$  such that

$$\omega_i \omega_j = \sum_{k=1}^n c_{ijk} \omega_k.$$

Generally, an ideal  $I$  in Problems 3.1 and 3.2 can be given as  $I = (\alpha, \beta)$ , where

$$\alpha = \sum_{i=1}^n a_i \omega_i, \quad \beta = \sum_{i=1}^n b_i \omega_i$$

and  $a_i, b_i \in \mathbb{Z}$ ,  $1 \leq i \leq n$ .

Before answering the above problems, we introduce a crucial definition about finite rings.

**Definition 3.3.** *Let  $R$  be a finite ring, a basis representation [10, 11] of  $R$  is a sequence of integers  $(m; d_1, \dots, d_m; (l_{ijk})_{i,j,k=1,\dots,m})$ , where  $m > 0$ ,  $d_i \geq 2$  and  $0 \leq l_{ijk} < d_k$ , such that*

- (1) *the additive group  $(R, +) = \mathbb{Z}_{d_1} v_1 \oplus \dots \oplus \mathbb{Z}_{d_m} v_m$ , where  $d_i$  are the additive orders of generator  $v_i$ , and*
- (2) *the multiplication of  $\{v_1, \dots, v_m\}$  is given by*

$$v_i v_j = \sum_{k=1}^m l_{ijk} v_k.$$

*Integers  $l_{ijk}$  are called structure constants.*

**Remark 3.4.** (1) *Using a basis representation, elements of  $R$  are encoded as vectors  $(x_1, \dots, x_m)$  with  $0 \leq x_i < d_i$ . Addition is componentwise and multiplication is defined by the structure constants, that is*

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_m) = \left( \sum_{i,j=1}^m x_i y_j l_{ij1} \bmod d_1, \dots, \sum_{i,j=1}^m x_i y_j l_{ijm} \bmod d_m \right).$$

(2) *The basis representation of a finite ring  $R$  is not unique. Any basis representation of  $R$  has length  $O(\log^3 |R|)$  bits (see [15]).*

We start with an auxiliary algorithm to pre-compute a number  $h$  that is a multiple of  $\mathcal{N}(\alpha)$  and  $\mathcal{N}(\beta)$ .

**Lemma 3.5.** *Let  $\mathcal{O}$  and  $I$  be as Problem 3.1. Then there exists a deterministic algorithm to output a positive integer  $h$ , such that  $h$  is a multiple of  $\mathcal{N}(\alpha)$  and  $\mathcal{N}(\beta)$ . The algorithm runs in  $O(n^3 B(n \log n T))$  bit operations, if all integers  $c_{ijk}$ ,  $a_i$ ,  $b_i$ ,  $1 \leq i, j, k \leq n$  are bounded in absolute value by  $T$ .*

*Proof* In Problem 3.1 the ideals

$$\mathcal{O}\alpha = \mathbb{Z}\alpha\omega_1 \oplus \dots \oplus \mathbb{Z}\alpha\omega_n, \quad \mathcal{O}\beta = \mathbb{Z}\beta\omega_1 \oplus \dots \oplus \mathbb{Z}\beta\omega_n.$$

We will compute two  $n \times n$  integral matrices  $A = (a_{ij})$ ,  $B = (b_{ij})$  such that

$$(\alpha\omega_1, \dots, \alpha\omega_n) = (\omega_1, \dots, \omega_n)A^T, \quad (\beta\omega_1, \dots, \beta\omega_n) = (\omega_1, \dots, \omega_n)B^T.$$

With the application of the multiplication table of  $\mathcal{W}$ , we have

$$\alpha\omega_i = \sum_{j=1}^n a_j\omega_i\omega_j = \sum_{j=1}^n \left( \sum_{k=1}^n a_k c_{kij} \right) \omega_j$$

$$\beta\omega_i = \sum_{j=1}^n b_j\omega_i\omega_j = \sum_{j=1}^n \left( \sum_{k=1}^n b_k c_{kij} \right) \omega_j.$$

Thus we obtain

$$a_{ij} = \sum_{k=1}^n a_k c_{kij}, \quad b_{ij} = \sum_{k=1}^n b_k c_{kij}$$

for all  $1 \leq i, j \leq n$ . Moreover  $|a_{ij}|, |b_{ij}| \leq nT^2$ . We can take  $h = |\det(AB)| = \mathcal{N}(\alpha\beta)$ , which is the required number. It is easy to see that the entries of  $AB$  are bounded in absolute value by  $n^3T^4$ . By the Hadamard inequality, we get  $h \leq n^{n/2}(n^3T^4)^n = n^{7n/2}T^{4n}$ .

Computing matrices  $A$  and  $B$  can be done in  $O(n^3M(\log nT))$  bit operations. We can use small primes modular computation to compute the determinants of  $A$  and  $B$ . First apply Gaussian elimination to compute the determinants of  $A$  and  $B$  modulo small primes  $p$  no more than  $t = O(n \log nT)$ , then recover  $|\det(A)|$  and  $|\det(B)|$  by the Chinese remainder theorem (see [6]). It costs  $O(n^3B(n \log nT))$  bit operations to obtain the value of  $h$ . Hence the total complexity is  $O(n^3B(n \log nT))$  bit operations.  $\square$

Since  $\mathcal{O}\alpha \subseteq I$  and  $\mathcal{N}(I)|\mathcal{N}(\alpha)$ ,  $h$  is also a multiple of  $\mathcal{N}(I)$ . The core of our algorithms to settle Problems 3.1 and 3.2 comes next.

**Theorem 3.6.** *Let  $\mathcal{O}$  and  $I$  be as Problem 3.1. Let  $h$  be a positive integer that is a multiple of  $\mathcal{N}(\alpha)$  and  $\mathcal{N}(\beta)$ . Then there exists a deterministic algorithm (Algorithm OUTPUT – BASIS) that outputs either  $I = \mathcal{O}$  or a basis representation of  $\mathcal{O}/I$ . The running time of the algorithm is  $O(n^3B(\log nT) + n^4B(\log h)\log h)$  bit operations, where  $T$  is as Lemma 3.5.*

*Proof* We present Algorithm OUTPUT – BASIS as follows:

First we compute the HNF of  $A^T$  and  $B^T$  which are the ones in Lemma 3.5. Applying Proposition 2.5 to matrices  $A^T$  and  $B^T$ , it takes  $O(n^2B(\log nT) + n^3B(\log h))$  bit operations to obtain their HNF  $H_A$  and  $H_B$ . Then we have

$$\mathcal{O}\alpha = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n, \quad \mathcal{O}\beta = \mathbb{Z}\beta_1 \oplus \dots \oplus \mathbb{Z}\beta_n$$

where

$$(\alpha_1, \dots, \alpha_n) = (\omega_1, \dots, \omega_n)H_A, \quad (\beta_1, \dots, \beta_n) = (\omega_1, \dots, \omega_n)H_B$$

are the HNF basis of ideals  $\mathcal{O}\alpha$  and  $\mathcal{O}\beta$  with respect to  $\mathcal{W}$ .

Next we will compute the HNF basis of  $I$  with respect to  $\mathcal{W}$ . Denote the  $n \times 2n$  integral matrix  $(H_A \ H_B)$  by  $M$ . The columns of  $M$ , treated as the ordinates

representation with respect to  $\mathcal{W}$ , generate the ideal  $I$  because of  $I = \mathcal{O}\alpha + \mathcal{O}\beta$ . Similarly, we apply Proposition 2.5 to  $M$ , and obtain the HNF  $\tilde{H}_M$  of  $M$  in  $O(n^3B(\log h))$  bit operations since the entries of  $M$  are bounded in absolute value by  $h$ . Thereby the HNF basis of  $I$  is given by  $(\gamma_1, \dots, \gamma_n) = (\omega_1, \dots, \omega_n)H_M$ , where  $H_M$  is the nonzero  $n \times n$  upper triangular matrix of  $\tilde{H}_M = (0 \ H_M)$ , and  $\det(H_M) = \mathcal{N}(I) \neq 0$ .

An application of Proposition 2.6 to  $H_M$ , we obtain the SNF  $S$  of  $H_M$  and the transforming matrices  $U, V$  in  $O(n^3B(\log h)\log h)$  bit operations, where  $S = VH_MU$ . As in Proposition 2.7, we denote a vector  $(\eta_1, \dots, \eta_n) = (\omega_1, \dots, \omega_n)V^{-1}$ , so

$$(\gamma_1, \dots, \gamma_n)U = (\eta_1, \dots, \eta_n)S = (d_1\eta_1, \dots, d_n\eta_n)$$

where  $d_{i+1}|d_i$ ,  $1 \leq i < n$ , and  $\mathcal{N}(I) = \det(S) = \prod_{k=1}^n d_k$ . If  $\mathcal{N}(I) = 1$ , then Algorithm OUTPUT – BASIS produces as output  $I = \mathcal{O}$  and stop. Otherwise, we proceed with the algorithm in the following way. As Proposition 2.7

$$\mathcal{O} = \mathbb{Z}\eta_1 \oplus \dots \oplus \mathbb{Z}\eta_n, \quad I = \mathbb{Z}(d_1\eta_1) \oplus \dots \oplus \mathbb{Z}(d_n\eta_n).$$

It yields that  $(\mathcal{O}/I, +) = \mathbb{Z}_{d_1}\bar{\eta}_1 \oplus \dots \oplus \mathbb{Z}_{d_n}\bar{\eta}_n$ , where  $\bar{\eta}_i$  denotes the coset  $\eta_i + I$  belonged to the factor ring  $\mathcal{O}/I$ ,  $i = 1, \dots, n$ . To obtain a basis representation of  $\mathcal{O}/I$ , it suffices to compute the structure constants, denoted by  $l_{ijk}$ ,  $1 \leq i, j, k \leq n$ , such that

$$\bar{\eta}_i\bar{\eta}_j = \sum_{k=1}^n l_{ijk}\bar{\eta}_k, \quad 0 \leq l_{ijk} < d_k.$$

Since  $\omega_i\omega_j = \sum_{k=1}^n c_{ijk}\omega_k$  and  $(\eta_1, \dots, \eta_n) = (\omega_1, \dots, \omega_n)V^{-1}$ , after some computations we achieve an expression on  $n \times n$  matrices

$$(\eta_i\eta_j)_{1 \leq i, j \leq n} = (V^{-1})^T(\omega_i\omega_j)_{1 \leq i, j \leq n}V^{-1}.$$

Suppose  $\eta_i\eta_j = \sum_{k=1}^n t_{ijk}\eta_k$ ,  $1 \leq i, j \leq n$ , where  $t_{ijk} \in \mathbb{Z}$ . Denote  $n \times n$  integer matrices

$$A_k = (a_{ijk})_{1 \leq i, j \leq n} = (V^{-1})^T(c_{ijk})_{1 \leq i, j \leq n}V^{-1}, \quad 1 \leq k \leq n \quad (1)$$

Then one can verify that

$$\begin{pmatrix} t_{ij1} \\ \vdots \\ t_{ijn} \end{pmatrix} = V \begin{pmatrix} a_{ij1} \\ \vdots \\ a_{ijn} \end{pmatrix} \quad (2)$$

for all  $1 \leq i, j \leq n$ . Let  $\pi$  be the natural ring homomorphism from  $\mathbb{Z}$  to  $\mathbb{Z}_h$  by  $\pi(a) = a \bmod h$ ,  $\pi_k$  be the natural ring homomorphism from  $\mathbb{Z}$  to  $\mathbb{Z}_{d_k}$  by  $\pi_k(a) = a \bmod d_k$ ,  $1 \leq k \leq n$ . Since  $\prod_{k=1}^n d_k = \mathcal{N}(I)|h$ , there is a commutative diagram linked  $\pi$  to  $\pi_k$ :

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\pi_k} & \mathbb{Z}_{d_k} \\ \pi \downarrow & \nearrow \phi_k & \\ \mathbb{Z}_h & & \end{array}$$



where  $\phi_k$  is the natural ring homomorphism from  $\mathbb{Z}_h$  to  $\mathbb{Z}_{d_k}$  by  $\phi_k(a \bmod h) = a \bmod d_k$ . Here  $a \bmod x$  denotes the smallest nonnegative residue of  $a$  modulo  $x$  for  $x = h$  or  $d_k$ .

In the factor ring  $\mathcal{O}/I$  we have

$$\bar{\eta}_i \bar{\eta}_j = \overline{\eta_i \eta_j} = \overline{\sum_{k=1}^n t_{ijk} \eta_k} = \sum_{k=1}^n \pi_k(t_{ijk}) \bar{\eta}_k.$$

Hence it suffices to compute all values of  $\pi_k(t_{ijk})$  in order to obtain the required structure constants. Since  $\det(V) = \pm 1$ , one can perform row reductions on  $V$  to compute the inverse matrix  $\tilde{V}$  of  $V$  over the ring  $\mathbb{Z}_h$ , which can be done in  $O(n^3 \log^2 h)$  bit operations. Note that the matrix  $\tilde{V}$  need not be  $V^{-1}$  but we have  $\pi(V^{-1}) = \tilde{V}$ , where  $\pi$  acts on a matrix by mapping on each entry of it. Thus we can obtain all values of  $\pi(t_{ijk})$ ,  $1 \leq i, j, k \leq n$ , by computing the multiplication of matrices over  $\mathbb{Z}_h$  in (1) and (2). Then calculating all values of  $\pi_k(t_{ijk}) = \phi_k \circ \pi(t_{ijk})$ ,  $1 \leq i, j, k \leq n$ , can be done in  $O(n^4 M(\log h) + n^3 \log^2 h) = O(n^4 \log^2 h)$  bit operations.

Finally, we can take  $l_{ijk} = \pi_k(t_{ijk})$ , and output those integers  $d_i$  and  $l_{ijk}$ ,  $1 \leq i, j, k \leq m$ , such that all  $d_1, \dots, d_m$  are greater than 1, where  $1 \leq m \leq n$ . That is, Algorithm OUTPUT – BASIS outputs a basis representation  $(m; d_1, \dots, d_m; (l_{ijk})_{i,j,k=1,\dots,m})$  of  $\mathcal{O}/I$ , where  $m > 0$ ,  $d_i \geq 2$  and  $0 \leq l_{ijk} < d_k$ . And the total complexity of Algorithm OUTPUT – BASIS is  $O(n^3 B(\log n T) + n^4 B(\log h) \log h)$  bit operations.  $\square$

**Remark 3.7.** *Under the pre-computation of the number  $h$ , we find that Algorithm OUTPUT – BASIS is quite practical, especially when we receive a smaller  $h$ . Maybe we know such  $h$  in advance, thus the algorithm of Lemma 3.5 could be omitted.*

The availability of a basis representation of  $\mathcal{O}/I$  is very crucial. According to Proposition 2.3 (resp. Proposition 2.4), it suffices to determine whether  $\mathcal{O}/I$  is a field (resp. a local ring) or not for solving Problem 3.1 (resp. Problem 3.2). Arvind et al [2] stated that field testing of finite rings in a basis representation is in P. We are now in a position to describe their algorithm (i.e. Algorithm 1) in details, and consider its computational complexity.

**Theorem 3.8.** *Algorithm 1 returns TRUE if and only if the finite ring  $R$  is a field and runs in  $O(M(\log^{15/2} p) + m^6 \log^3 p)$  bit operations, where  $p = \min\{d_1, \dots, d_m\}$ .*

*Proof* Correctness follows from [2]. We proceed with the proof of the running time. In line 1, the complexity is dominated by any known bound for deterministic primality testing. Therefore it takes  $O(M(\log^{15/2} p))$  bit operations by the AKS test [1]. And computing the minimal polynomial of  $v_1$  in line 5 can be done in  $O(m^\omega \log m \log^2 p)$  bit operations. Indeed,  $(R, +) = \mathbb{F}_p v_1 \oplus \dots \oplus \mathbb{F}_p v_m$  is a  $\mathbb{F}_p$ -algebra. Applying the method of [15] to  $R$ , we obtain the above complexity. It takes  $O(m^{(\omega+1)/2} \log m \log \log m \log^3 p)$  bit operations to determine whether  $f_1$  is reducible in line 7 (see [13]).

---

Algorithm 1: IS-FIELD

---

**Input:**

a finite ring  $R$  given by a basis representation  $(m; d_1, \dots, d_m; (l_{ijk})_{i,j,k=1,\dots,m})$

**Output:**

TRUE if  $R$  is a field, FALSE otherwise

- 1: If  $d_1 = \dots = d_m$  is prime, GOTO line 3 ; Otherwise
- 2: **return** FALSE
- 3: If  $m = 1$ , then
- 4: **return** TRUE
- 5:  $p \leftarrow d_1$ ,  $f_1 \leftarrow$  the minimal polynomial of the first generator  $v_1$  over  $\mathbb{F}_p$
- 6:  $m_1 \leftarrow$  the degree of  $f_1$
- 7: If  $f_1$  is reducible over  $\mathbb{F}_p$ , then
- 8: **return** FALSE
- 9: If  $m_1 = m$ , then
- 10: **return** TRUE
- 11: **for**  $i = 2$  to  $m$  **do**
- 12:    $f_i \leftarrow$  the minimal polynomial of  $i$ -th generator  $v_i$  over  $\mathbb{F}_p(v_1, \dots, v_{i-1})$
- 13:    $m_i \leftarrow$  the degree of  $f_i$
- 14:   If  $f_i$  is reducible over  $\mathbb{F}_p(v_1, \dots, v_{i-1})$ , then
- 15:   **return** FALSE
- 16:   If  $\prod_{j=1}^i m_j = m$ , then
- 17:   **return** TRUE
- 18: **end for**

---

Next we will state a method to compute the minimal polynomial  $f_i$  of  $v_i$  over the field  $F_{i-1} = \mathbb{F}_p(v_1, \dots, v_{i-1})$ , where  $i > 1$ . It takes  $O(m^\omega \log m \log^2 p)$  bit operations to determine a matrix  $E \in M_{(m+1) \times m}(\mathbb{F}_p)$  such that

$$(1, v_i, \dots, v_i^m) = (v_1, \dots, v_m) E^T. \quad (3)$$

The same technique as [15] is used. Then we compute a great linearly independent subset  $S$  of  $\{v_1, \dots, v_m\}$  over  $F_{i-1}$  in the following way. For instance, computing a great linearly independent subset of  $\{v_1, v_2\}$  is equivalent to solving the equation (4) of variables  $x$  and  $y$  belonged to  $F_{i-1}$ :

$$xv_1 + yv_2 = 0 \quad (4)$$

Since  $\{v_1^{t_1} \dots v_{i-1}^{t_{i-1}} \mid 0 \leq t_j < m_j, \text{ for } 1 \leq j < i\}$  is a  $\mathbb{F}_p$ -basis of  $F_{i-1}$ , we can write  $x$  and  $y$  in the coordinate representation to this  $\mathbb{F}_p$ -basis. Then (4) is converted into a linear system of equations over  $\mathbb{F}_p$  with the help of the known structures of  $F_{i-1}$  and  $R$ . The system of equations owns  $m$  equations and  $2 \prod_{j=1}^{i-1} m_j$  variables, where  $\prod_{j=1}^{i-1} m_j \leq m$ . It can

be solved by performing  $2m^\omega$  operations in  $\mathbb{F}_p$ . We repeat this procedure for  $m-1$  steps by adding all generators  $\{v_2, \dots, v_m\}$  one at a time to  $v_1$ , then  $S$  can be computed in  $(2 + \dots + m)m^\omega \log^2 p = O(m^{\omega+2} \log^2 p)$  bit operations. We may assume  $S = \{\mu_1, \dots, \mu_s\}$ . As a by-product one simultaneously receives a matrix  $H \in M_{m \times s}(F_{i-1})$  such that

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = H \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_s \end{pmatrix} \quad (5)$$

Combining (3) with (5) we get

$$\begin{pmatrix} 1 \\ v_i \\ \vdots \\ v_i^m \end{pmatrix} = EH \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_s \end{pmatrix} \quad (6)$$

Then one can perform row reductions on the matrix  $EH$  to gain the minimal polynomial  $f_i$  of  $v_i$  over  $F_{i-1}$ . Hence computing  $f_i$  can be done in  $O(m^{\omega+2} \log^2 p)$  bit operations. And it costs  $O(m_i^{(\omega+1)/2} m^3 \log m \log \log m \log^3 p)$  bit operations for line 14 applying the method of [13], where  $m_i \leq m$ . Finally, the total complexity of Algorithm 1 is  $O(M(\log^{15/2} p) + m^6 \log^3 p)$  bit operations, where  $p = \min\{d_1, \dots, d_m\}$ .  $\square$

As for local ring testing, Staromiejski [15] presented a deterministic polynomial-time algorithm, which is given as Proposition 3.9 below.

**Proposition 3.9.** *There exists a deterministic algorithm (Algorithm IS – LOCAL) that receives as input a finite ring  $R$  in a basis representation  $(m; d_1, \dots, d_m; (l_{ijk})_{i,j,k=1,\dots,m})$ , and produces as output  $R$  is a local ring or not. The running time of the algorithm is  $O(M(\log^{15/2} p) + \log^4 |R|)$  bit operations, where  $p = \min\{d_1, \dots, d_m\}$ .*

**Remark 3.10.** (1) Since  $p^m \leq |R|$ , we have  $m^6 \log^3 p \leq \log^6 |R|$  and  $M(\log^{15/2} p) \leq \log^{17/2} |R|$ . That is to say, both Algorithm 1 and Algorithm IS – LOCAL are polynomial time in the basis representation size.

(2) Algorithm 1 and Algorithm IS – LOCAL have a common feature, which is the deterministic AKS test [1] dominated their computational complexity in the case of a large  $p$ . Hence we can employ a randomized primality test (e.g. the Miller-Rabin test) in Algorithm 1 to obtain an efficient randomized algorithm, as [15] has done.

With the above preparations, we can now answer the beginning Problems 3.1 and 3.2 in the following way.

**Proposition 3.11.** *Algorithm 2 returns TRUE if and only if  $I$  is a prime ideal of  $\mathcal{O}$  and performs in  $O(M(\log^{15/2} h) + n^3 B(\log n T) + n^4 B(\log h) \log h)$  bit operations, if all integers  $c_{ijk}$ ,  $a_i$ ,  $b_i$ ,  $1 \leq i, j, k \leq n$ , are bounded in absolute value by  $T$ .*

---

Algorithm 2: IS-PRIME-IDEAL

---

**Input:**

a multiplication table  $((c_{ijk})_{1 \leq i,j,k \leq n})$  of  $\mathcal{W}$  to a Dedekind domain  $\mathcal{O}$  of finite rank;  
a nonzero ideal  $I = (\alpha, \beta)$  with  $(a_1, \dots, a_n)$ ,  $(b_1, \dots, b_n)$  are the coordinates of  $\alpha$ ,  $\beta$  with respect to  $\mathcal{W}$ ;  
a positive integer  $h$  such that  $\text{lcm}(\mathcal{N}(\alpha), \mathcal{N}(\beta)) | h$

**Output:**

TRUE if  $I$  is a prime ideal, FALSE otherwise  
1:  $q \leftarrow \mathcal{N}(I)$   
2: If  $q > 1$ , GOTO line 4; Otherwise  
3: **return** FALSE  
4: Compute a basis representation  $(m; d_1, \dots, d_m; (l_{ijk})_{i,j,k=1,\dots,m})$  of  $\mathcal{O}/I$   
5: **return** IS-FIELD( $\mathcal{O}/I$ )

---

*Proof* Correctness easily follows from Proposition 2.3 and Theorem 3.8. Since  $|\mathcal{O}/I| = d_1 \cdot \dots \cdot d_m | h$ , the running time of Algorithm 2 can be deduced from Theorems 3.6 and 3.8 directly.  $\square$

---

Algorithm 3: IS-PRIME-IDEAL-POWER

---

**Input:**

a multiplication table  $((c_{ijk})_{1 \leq i,j,k \leq n})$  of  $\mathcal{W}$  to a Dedekind domain  $\mathcal{O}$  of finite rank;  
a nonzero ideal  $I = (\alpha, \beta)$  with  $(a_1, \dots, a_n)$ ,  $(b_1, \dots, b_n)$  are the coordinates of  $\alpha$ ,  $\beta$  with respect to  $\mathcal{W}$ ;  
a positive integer  $h$  such that  $\text{lcm}(\mathcal{N}(\alpha), \mathcal{N}(\beta)) | h$

**Output:**

TRUE if  $I$  is a prime ideal power, FALSE otherwise  
1:  $q \leftarrow \mathcal{N}(I)$   
2: If  $q > 1$ , GOTO line 4; Otherwise  
3: **return** FALSE  
4: Compute a basis representation  $(m; d_1, \dots, d_m; (l_{ijk})_{i,j,k=1,\dots,m})$  of  $\mathcal{O}/I$   
5: **return** IS-LOCAL( $\mathcal{O}/I$ )

---

**Proposition 3.12.** *Algorithm 3 returns TRUE if and only if  $I$  is a power of a prime ideal of  $\mathcal{O}$ . It runs in time  $O(M(\log^{15/2} h) + n^3 B(\log n T) + n^4 B(\log h) \log h)$ , where  $T$  is as before.*

*Proof* Correctness follows from Propositions 2.4 and 3.9 immediately. The total complexity is obtained the same as Proposition 3.11.  $\square$

**Remark 3.13.** *The input size of Algorithms 2 and 3 is  $O(n^3 \log T)$  bits. One can verify that Algorithms 2 and 3 are polynomial time in the input size. Actually, we can take*

$h = \mathcal{N}(\alpha\beta) \leq n^{7n/2}T^{4n}$  by Lemma 3.5, i.e.  $\log h = O(n \log n T)$ , and obtain the asserted polynomial-time.

## 4 Conclusions

We presented a deterministic polynomial-time algorithm of testing whether a nonzero ideal is a prime ideal (or a prime ideal power) in a Dedekind domain with finite rank in Section 3. There is one benefit of the number  $h$  in these algorithms, that is, a smaller  $h$  leads to the prime ideal test (or prime ideal power test) more efficient. Furthermore, if we employ a randomized primality test and receive a small value of  $h$  in advance, then it gives rise to a randomized but very efficient test.

If an integral basis of the ring of algebraic integers of a number field has been computed, then a deterministic polynomial-time algorithm for testing the primality of ideals in this ring can be derived from our prime ideal test. This is because the algebraic integers ring is a Dedekind domain of finite rank. It is natural to ask whether there exists a deterministic polynomial-time algorithm for testing the primality of ideals in a general Dedekind domain, not necessarily of finite rank. At this time, the corresponding factor ring need not be a finite ring, such as  $\mathcal{O} = \mathbb{Q}[X]$ ,  $I = (X)$  and  $\mathcal{O}/I \cong \mathbb{Q}$  is infinite. Hence the current method based on basis representation will not work any more. We are looking forward to finding a new method.

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